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BRS symmetry in Connes' non-commutative geometry

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Abstract. We extend the BRS and anti-BRS symmetry to the two-point space of Connes' non-commutative model building scheme. The constraint relations are derived and the quantum Lagrangian constructed. We find that the quantum Lagrangian can be written as a functional of the curvature for symmetric gauges with the BRS, anti-BRS auxiliary field finding a geometrical interepretation as the extension of the Higgs scalar.

1. Introduction

Non-commutative geometry has emerged as a promising model building prescription providing a possible underlying structure to the appearance of Higgs scalars with point-like interactions and quartic curvatures. As the name suggests, this is achieved by generalizing the underlying notion of geometry as applied to particle physics. This generalization is supported by the Gelfand–Naimark theory: let $\mathcal{A}$ be a C^*-algebra with unity, then if $\mathcal{A}$ is commutative it will be isomorphic to the algebra of all continuous complex-valued functions $C^\infty(X)$, defined on a compact topological space, X. Thus, rather than the manifold itself, one addresses the algebra of smooth functions defined over it, which is equivalent. A differential calculus is then constructed on $C^\infty(X)$. This provides a very convenient way to generalize the topological space, X, in a sense linearizing the description of complicated or 'badly behaved' spaces.

There is more than one approach to implementing this geometric view. One perspective, developed by Dubois-Violette, Kerner and Modore [1], extends the set of complex smooth functions over space-time to include complex matrices, i.e. $C^\infty(M) \to C^\infty(M) \otimes M_n(\mathrm{C})$; M being space-time and $M_n(\mathrm{C})$ the set of $n \times n$ complex matrices [1]. It is in this sense that the geometric prescription is non-commutative; see also Balakrishna *et al* [2]. The important aspect here is that the differential calculus is developed on the entire algebra so one does not simply yield matrix-valued forms. Indeed Higgs scalars emerge as generalized 1-forms valued in the derivation algebra of $M_n(\mathrm{C})$. This is reminiscent of BRS analysis in normal gauge theory, indicating perhaps a deeper underlying assosciation.

The non-commutative model building scheme which will be of particular interest in this paper is that developed by Connes [3]; see also Connes and Lott [4, 5] as well as Chamseddine *et al* [6] who reformulated this approach to include GUT models. Connes generalized the geometric prescription by extending the algebra of smooth functions to include a two-point space, thus e.g. $C^\infty(M) \to C^\infty(M) \oplus C^\infty(M)$. Gauge fields arise as appropriately defined fibre bundles on each copy of space-time, while Higgs scalars appear as connections between these copies. In this way the symmetry breaking scale receives a geometrical interpretation, being just the inverse distance between space-times. This construction places severe restrictions on the models which can be constructed. For instance, flavour chirality is essential. Furthermore, survival of the Higgs potential demands the existence of multiple, non-degenerate, fermionic families. This is an intriguing correspondence with phenomenology [7]. It also appears that constraints upon the Higgs and top quark masses appear at the classical level [8]. This is a natural consequence of entering fermionic data into the model as a starting point towards reproducing observed behaviour.

The standard model constructed by Connes and Lott is a remarkable success of this model building prescription [5]. Nevertheless, a quantum theory is still lacking. This state of affairs is not unreasonable given the intrinsically new setting of this approach. Indeed connections with quantum theory are now emerging, perhaps suggesting that the non-commutative settings are fundamentally quantum mechanical [9]. On a less ambitious level, however, it appears that the constraints imposed by non-commutative geometry do not survive quantum corrections [10]. This is a common symptom of such 'Kaluza–Klein-like' model building schemes (see also the connection with coset space dimensional reduction [11]). However, given the novelty of this geometric description, it is not unreasonable that an additional symmetry or some more exotic mechanism exists in which quantum corrections are consistent [12]. In this context an intrinsic quantum mechanical connection is indeed compelling.

The failure of the non-commutative model building constraints to survive quantum corrections was demonstrated in only the simplest possible Abelian model (the standard model evolution was shown, tentatively, to be slow [10]). Given that the quantum connection has not, nevertheless, been satisfactorily resolved we wish, in this paper, to extend to non-Abelian models and consider the non-commutative implementation of BRS symmetry in Connes' model building scheme. Independently of providing a possible framework for quantizing such models, we seek to generalize the gauge-fixing mechanism at the classical level. This is made possible by the geometric origin of the BRS and anti-BRS constraints [13]. Since in non-commutative geometry the Higgs scalar appears on the same level as gauge fields we have a conceptually simple means by which matter fields may be included in the notion of BRS symmetry. This is compelling if one recalls the important role that Higgs scalars play in interesting soultions of Yang–Mills fields, such as monoploes. We find that the extended geometric setting of non-commutative geometry provides a framework in which a unified description may be derived with a more adequate geometrical interpretation of the BRS/anti-BRS scalar emerging.

2. Non-commutative gauge theory

We will briefly overview the model building prescription of non-commutative gauge theory to set the mathematical formalism. While new perspectives to quantum theory are expanding the motivation for this approach [9, 11] the original motivation was geometric and we shall introduce the scheme in this setting.

To generalize the Riemannian metric the notion of geodesic distance must be consistently incorporated and this is encoded in the concept of a K-cycle. A K-cycle over the involutive algebra, $\mathcal{B}$ say, is a $*$-action of $\mathcal{B}$ by bounded operators on a Hilbert space $\mathcal{H}$, denoted by ρ, and a possibly unbounded, self-adjoint, operator D, denoted Dirac operator, such that $[D, \rho(f)]$ is a bounded operator $\forall f \in \mathcal{B}$ and $(1+D^2)^{-1}$ is compact. Let X be a compact Riemannian spin manifold, $\mathcal{F}$ the algebra of functions on X and $(\mathcal{H}_{\mathcal{F}}, \not\partial)$ the Dirac K-cycle with $\mathcal{H}_{\mathcal{F}} = L^2(x, \sqrt{g}\mathrm{d}^{\mathrm{d}}x)$ of $\mathcal{F}$. Denote by γ_5 the fifth anticommuting Dirac gamma matrix, the chirality operator, defining a Z_2 grading on $\mathcal{H}_{\mathcal{F}}$. Similarly a K-

cycle can be defined on the discrete set representing the internal space. Let $\mathcal{A}$ be given by $\mathcal{A} = M_n(\mathbf{C}) \oplus M_p(\mathbf{C}) \oplus \ldots$ corresponding to the Hilbert spaces $\mathbf{C}^n, \mathbf{C}^p, \ldots$, respectively. (Alternatively, one may define an n-point space $\mathcal{A} = C^\infty(X) \oplus C^\infty(X) \oplus \ldots$ and introduce an appropriate vector bundle $\mathcal{E} = e\mathcal{A}^p$, $p \in \mathbf{Z}$, where e is a projection. The algebra $\mathcal{A}$ and the vector bundle $\mathcal{E}$ then act together to define the required gauge groups on each copy of space-time [5]). For a two-point space the Dirac operator takes the form

$$D = \begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} \tag{1}$$

where M is a mass matrix of size $\dim \mathcal{H}_L \times \dim \mathcal{H}_R$, where $\mathcal{H} = \mathcal{H}_L \oplus \mathcal{H}_R$ decomposes under the action of a suitable chirality operator

$$\chi = \begin{pmatrix} 1_L & 0 \\ 0 & -1_R \end{pmatrix} . \tag{2}$$

For a product geometry $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}$ a product K-cycle is naturally defined with the generalized Dirac operator

$$D_t = \not{\partial} \otimes 1 + \gamma_5 \otimes D . \tag{3}$$

For any C^*-algebra a K-cycle will define a metric d on the state space of $\mathcal{B}$ by

$$d(p, q) = \sup\{|p(f) - q(f)| : f \in \mathcal{B}, ||[D, f]|| \leqslant 1\} . \tag{4}$$

Recall that now the points p and q are states on the algebra so that $p(f) \equiv f(p)$.

To make contact with gauge theories a differential algebra must be constructed on $\mathcal{B}$. The space of all differential forms $\hat{\Omega}^*(\mathcal{B}) = \bigoplus_{p \in N} \hat{\Omega}^p(\mathcal{B})$ is a graded differential algebra equipped with a differential operator δ such that

$$\delta : \hat{\Omega}^p(\mathcal{B}) \to \hat{\Omega}^{p+1}(\mathcal{B}) \tag{5}$$

along with nilpotency

$$\delta^2 = 0 . \tag{6}$$

The space of p-forms $\hat{\Omega}^p(\mathcal{B})$ is generated by finite sums of terms of the form

$$\hat{\Omega}^p(\mathcal{B}) = \left\{ \sum_j a_0^j \delta a_1^j \ldots \delta a_p^j, a_p^j \in \mathcal{B} \right\} \tag{7}$$

which follows from the relations

$$\delta 1 = 0 \qquad\qquad \delta(ab) = (\delta a)b + a\delta b \tag{8}$$

with the differential δ defined by

$$\delta(a_0 \delta a_1 \ldots \delta a_p) = \delta a_0 \delta a_1 \ldots \delta a_p . \tag{9}$$

Extending the representation of $\mathcal{B}$ on $\mathcal{H}$ to its universal differential envelope $\hat{\Omega}^*(\mathcal{B})$ is achieved via the map

$$\pi : \hat{\Omega}^*(\mathcal{B}) \to B(\mathcal{H}) \tag{10}$$

where $B(\mathcal{H})$ is the algebra of bounded operators on $\mathcal{H}$ defined by

$$\pi(a_0 \delta a_1 \ldots \delta a_p) = \rho(a_0)[D, \rho(a_1)] \ldots [D, \rho(a_p)] . \tag{11}$$

It is this representation on Hilbert space, as a way of connecting with our usual notions of space-time vectors and scalars, which distinguishes the Connes–Lott model building scheme [5]. The crucial aspect which must be considered, however, is that the representation π is ambiguous, with the correct space of forms actually given by

$$\Omega^*(\mathcal{B}) = \hat{\Omega}^*(\mathcal{B})/J \tag{12}$$

where J is given by the differential ideal [4, 7]

$$J = \ker\pi + \delta\ker\pi = \bigoplus_p J^p \tag{13}$$

where

$$J^p = (\ker\pi)^p + \delta(\ker\pi)^{p-1} . \tag{14}$$

Degree by degree the correct space of forms becomes

$$\Omega^0(\mathcal{B}) = \hat{\Omega}^0(\mathcal{B}) \cong \rho(\mathcal{B})$$
$$\Omega^1(\mathcal{B}) = \hat{\Omega}^1(\mathcal{B})/(\ker\pi)^1 \cong \pi(\hat{\Omega}^1(\mathcal{B}))$$

and for degree $p \geqslant 2$

$$\Omega^p(\mathcal{B}) \cong \pi(\hat{\Omega}^p(\mathcal{B}))/\pi(\delta(\ker\pi)^{p-1}) . \tag{15}$$

When $\mathcal{B} = \mathcal{F}$, $\Omega^*(\mathcal{B})$ recovers deRham's differential algebra of differential forms on X.

The simplest means by which the quotient may be considered as a subspace is in the prescence of a scalar product. This is naturally defined on the internal space by the trace on matrices. In the infinite-dimensional case the inner product is defined by the Dixmier trace

$$\mathrm{tr}_w(Q|\not{\partial}|^{-d}) = \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \lambda_n \tag{16}$$

where Q is a bound operator on $\mathcal{H}_{\mathcal{F}}$, $d = \dim X$ and λ_n are eigenvalues of $Q|\not{\partial}|^{-d}$ arranged in a decreasing sequence discarding the Dirac zero modes. By correspondence with the deRham complex on X this reduces to the usual scalar product on X

$$(\phi, \psi) = 1/8\pi^2 \int_X \phi^* * \psi \qquad \phi, \psi \in \Omega^p(X) \tag{17}$$

incorporating the Hodge star. Since the fermionic fields are the fundamental fields the spinor action can be simply written down.

3. The geometry of BRS symmetry

As is well known, difficulties arise when covariantly quantizing non-Abelian gauge theories from unwanted contributions to the gluon propogator. Transversality and unitarity are spoilt in closed-loop diagrams from the longitudinal part of the propogator. To overcome this it is first necessary to introduce a constant notion of transversality on the gauge field and this is implemented by gauge fixing (up to problems arising from the Gribov ambiguity). Since differing gauges need not be smoothly connected it becomes necessary to restrict to an appropriate region of configuration space by choosing a particular representative in some equivalence class of gauge related connections. In the functional integral representation the Jacobian of this gauge-fixing term can be written in terms of a set of fermionic scalars, known as ghosts [14]. Since loop diagrams involving such ghosts will introduce factors of (-1) due to their fermionic nature they will ensure the cancellation of unitarity violating terms in the perturbation expansion. (In the Abelian case the Ward identities are satisfied without the need for ghosts). The gauge-fixing term results in the loss of gauge invariance for the new 'quantum Lagrangian' now consisting of the original terms plus the gauge fixing and ghost contributions. However, a new global symmetry can be defined with all the consequences of gauge invariance; this is the BRS symmetry.

The geometrical origin of the BRS (as well as anti-BRS) invariance was demonstrated by Baulieu and Thierry-Mieg by reversing the construction of Yang–Mills theories [13]. Gauge fields and ghosts are now introduced as the fundamental independent fields. This is motivated by the geometrical description of gauge theories in terms of principal fibre bundles, which consist of a base space (space-time with coordinates x) and fibres corresponding to local copies of the gauge group (with coordinates y). A matter field in this (x, y) space takes the form

$$\tilde{\psi}(x, y) = \exp(\mathrm{i}y^m T^\alpha_{m\beta})\psi^\beta(x) \tag{18}$$

where T_m are the generators of the Lie group. This describes our usual notion of gauge transformation. The extension is made when one generalizes the gauge field 1-form $A_\mu(x)\,\mathrm{d}x^\mu$ to the (x, y) space

$$\tilde{A}^a(x, y) = A^a_\mu(x, y)\mathrm{d}x^\mu + C^a_m(x, y)\mathrm{d}y^m \tag{19}$$

where $C^a_m(x, y)$ is a scalar field with $\mathrm{d}x^\mu$ and $\mathrm{d}y^m$ spanning the cotangent space of the fibre bundle. The ghost field is identified with $C^a_m(x, y)\,\mathrm{d}y^m$ which anticommute by virtue of being differential forms. The anti-BRS ghost field can be similarly introduced by constructing a 'double' principal bundle with coordinates $(x, y, \overline{y})$. This is isomorphic to the product of space-time by two copies of the gauge group. However, this is not to be confused with the two point space of Connes' construction [5] as there is no notion of geodesic distance between these copies. The generalized gauge field now becomes

$$\tilde{A}^a(x, y, \bar{y}) = A^a_\mu(x, y, \bar{y})\,\mathrm{d}x^\mu + C^a_m(x, y, \bar{y})\,\mathrm{d}y^m + \bar{C}^a_m(x, y, \bar{y})\,\mathrm{d}\bar{y}^m\,. \tag{20}$$

BRS and anti-BRS equations are now introduced as geometrical constraints arising from constructing the generalized curvature

$$\tilde{F}^a = \tilde{d}\tilde{A}^a + 1/2[\tilde{A}, \tilde{A}]^a\,. \tag{21}$$

Imposing the Cartan–Maurer condition insures compatability of the fibration with parallel transport, restricting the curvature to be proportional to $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu$. The remaing terms, such as $\mathrm{d}y^m \wedge \mathrm{d}x^\mu$, must cancel, thus yielding the required equations.

The association of this approach to the non-commutative model building scheme of Dubois-Violette *et al* [1] should now be clear. Indeed this was directly exploited by Balakrishna *et al* [2]. The difference lies in the interpretation of these new scalar fields as being bosonic or fermionic. Extending Connes' approach then will effectively constitute employing both model building directions; that of Connes [3–5] and that of Dubois-Violette *et al* [1]. This has, in fact, been used before to yield sufficiently diverse sets of Higgs scalars to accommodate required symmetry-breaking patterns [15]. Importantly, it appears that in exploring BRS symmetry we will be exploiting the notion of 'internal space' to its fullest limits in deriving a suitable model.

4. Application to parallel space-times

That interesting generalizations may emerge from applying these notions to Connes' construction is amply demonstrated by the richness of structure when cohomological considerations are directed to non-commutative geometry [16]. As the previous construction suggests, we wish to extend our algebraic considerations onto the group manifolds on each copy of space-time. This has already been explored by Watamura for quantum groups in the case of one space-time [17]. This sets the mathematical tone, where on each copy of space-time we will restrict ourselves to the classical groups, each extended copy now being described by the algebra

$$C^\infty(X \otimes G) \cong C^\infty(X) \otimes C^\infty(G) \tag{22}$$

the isomorphism arising due to the local triviality of the principal fibre bundle. Ghosts will now appear due to $C^\infty(G)$, G being a compact unitary group. Our analysis will differ in that a set of parallel space-times will also introduce scalar degrees of freedom as connections between these copies. This brings us to the interesting role that matter fields will now play in our analysis. The idea is very simple. As with other model building schemes which exploit 'internal' structure, the Higgs scalar and gauge fields are recognized as originating from a single underlying principal, i.e. they are unified. We propose then to extend the scalar contributions in an analogous manner to that of the gauge fields in (20). Thus

$$\tilde{\phi}(x, y, \bar{y}) = \phi(x, y, \bar{y}) + \beta(x, y, \bar{y}) + \bar{\beta}(x, y, \bar{y})\,. \tag{23}$$

The β fields correspond to connections between group manifolds analogously to ϕ, which is a connection between space-times. This should arise naturally when the underlying algebraic construction is represented on Hilbert space.

A subtle but important distinction now arises with the model building approach of Dubois-Violette *et al* [1]. Strictly they are considering an extension of the differential calculus to matrix algebras. However, we are considering the set of smooth functions over the group manifolds on each copy of space-time. In this sense the matrix structure serves as a local basis against which coordinates on the group manifolds are defined, i.e. on the tangent space to the group. We do not seek to construct a derivation algebra on the matrix algebra but rather on the coordinates defined locally by the Lie algebra.

To implement the extension we begin at the level of the algebra and generalize the differential operator δ to the set

$$\tilde{\delta} = (\delta, \delta_Q, \delta_{\bar{Q}}) \tag{24}$$

the subscripts Q and $\bar{Q}$ are introduced to signify the BRS and anti-BRS operators. As we are restricting ourselves to classical groups all matrix elements will be real or complex. We are thus strictly dealing with commutative Hopf algebras. The set $\tilde{\delta}$ satisfy the Leibnitz rule where, in order to satisfy nilpotency, the following are also implied

$$\delta^2 = \delta_Q^2 = \delta_{\bar{Q}}^2 = 0 \qquad \text{and} \qquad \{\delta, \delta_Q\} = \{\delta, \delta_{\bar{Q}}\} = \{\delta_Q, \delta_{\bar{Q}}\} = 0\,. \tag{25}$$

5. The BRS and anti-BRS constraints

Just as Baulieu *et al* [13] introduced ghosts as *a priori* geometrical fields we are considering the dynamical fields as existing on an extended notion of manifold described appropriately by a smooth algebra. We are thus considering matrix elements

$$u_{ik} \in C^\infty(X \otimes G) \cong C^\infty(X) \otimes C^\infty(G) \tag{26}$$

on each copy of space-time. Furthermore, the groups are taken to be compact. Thus we can be confident that our generalized connection will be a finite sum of the form

$$\tilde{\omega} = \sum_i a^i \tilde{\delta} b^i = \sum_i a^i (\delta + \delta_Q + \delta_{\bar{Q}}) b^i\,. \tag{27}$$

This generalizes our notion of 1-form on the algebra where $a^i, b^i \in \mathcal{A}_t$, $\mathcal{A}_t$ being the total algebra,

$$\mathcal{A}_t = C^\infty(X \otimes G_1) \oplus C^\infty(X \otimes G_2)\,. \tag{28}$$

Note that we do not assume that $G_1 = G_2$.

To represent this on Hilbert space we must extend the representation of Connes' [3–5] onto the group manifolds. This requires constructing a Clifford algebra on the compact internal spaces as extensions of the space-time Clifford algebra. Clearly there are parallels here with Kaluza–Klein theory. For an N-dimensional internal manifold this will correspond to introducing a Clifford algebra belonging to the group $O(N)$. This need not be associated with the underlying group upon which it is based. Compare this with the construction of Baulieu *et al* [13] which follows the same pattern. We need not consider this to be a problem when it is recalled that the Clifford algebra is introduced only as a means to represent differential forms. In this sense the physical fields are not valued in this algebra. Such a construction arises from demanding that the ghost fields carry both gauge *and* internal vector indices.

Corresponding to the differential operator $\tilde{\delta}$ we write down a Dirac operator

$$\begin{aligned}\tilde{D} &= D + Q + \bar{Q}\\ &= \begin{pmatrix} \partial\!\!\!/ & \gamma_5 M \\ \gamma_5 M^* & \partial\!\!\!/ \end{pmatrix} + \gamma_5 \otimes \begin{pmatrix} \partial\!\!\!/_m & \tilde{\gamma}_5 \xi \\ \tilde{\gamma}_5 \xi^* & \partial\!\!\!/_n \end{pmatrix} + \gamma_5 \otimes \begin{pmatrix} \bar{\partial\!\!\!/}_m & \tilde{\gamma}_5 \bar{\xi} \\ \tilde{\gamma}_5 \bar{\xi}^* & \bar{\partial\!\!\!/}_n \end{pmatrix}.\end{aligned} \tag{29}$$

Here γ_5 is the usual chirality operator on four-dimensional space-time, $\tilde{\gamma}_5$ is the corresponding operator on the group manifold, $\partial\!\!\!/_p$, $p = m, n$ are the operators on each group manifold while ξ is an $m \times n$ matrix yet to be specified. Strictly, since $\tilde{\gamma}_5$ anti-commutes with the Clifford algebras on both group manifolds we should write $\tilde{\gamma}_5 = \gamma_5^m \otimes \gamma_5^n$, which will be understood from now on. For clarity we can explicitly set out the K-cycles of the model. Writing the total algebra (28) as

$$C^\infty(X) \otimes (C^\infty(G_1) \oplus C^\infty(G_2)) \tag{30}$$

the generalization of our notion of internal space becomes clear. The Dirac operator on the continuous manifolds takes the form $\partial\!\!\!/ + \gamma_5(\partial\!\!\!/_p + \bar{\partial\!\!\!/}_p)$, operating on the space of square integrable functions on space-time and smooth functions on the group manifolds. The functions on the group manifolds have the form $u_{ik}(y)$ corresponding to matrix elements in the space of complex matrices $M_p(\mathbf{C})$. On the discrete space we introduce the Dirac operator

$$\begin{pmatrix} 0 & M \\ M^* & 0 \end{pmatrix} + \tilde{\gamma}_5 \left\{ \begin{pmatrix} 0 & \xi \\ \xi^* & 0 \end{pmatrix} + \begin{pmatrix} 0 & \bar{\xi} \\ \bar{\xi}^* & 0 \end{pmatrix} \right\} \tag{31}$$

acting on the Hilbert space $\mathbf{C}^m \oplus \mathbf{C}^n$. The extension of the internal Dirac operator introduces the notion of a connection between group manifolds analogously to that between space-times. The K-cycle for both continuous and discrete contributions is now described by the operator (29). (Note that the 'product' K-cycle is actually between the classical K-cycle for $C^\infty(X)$ and our generalized concept of internal space.) It is important to observe that since $O(N)$ has fundamental group Z_2 for all $N > 2$ there will be no problem in defining the notion of spinors on the continuous internal manifolds for non-Abelian groups.

Representing the connection 1-form on Hilbert space we have

$$\rho(a^i) = \begin{pmatrix} A_0^i & 0 \\ 0 & B_0^i \end{pmatrix} \qquad \rho(b^i) = \begin{pmatrix} A_1^i & o \\ 0 & B_1^i \end{pmatrix} \tag{32}$$

so that

$$
\begin{aligned}
\pi(\tilde{\omega}) &= \sum_i \rho(a^i)[D + Q + \bar{Q}, \rho(b^i)] \\
&= \sum_i \left\{ \begin{pmatrix} A_0^i \partial\!\!\!/ A_1^i & \gamma_5 A_0^i (M B_1^i - A_1^i M) \\ \gamma_5 B_0^i (M^* A_1^i - B_1^i M^*) & B_0^i \partial\!\!\!/ B_1^i \end{pmatrix} + \gamma_5 \otimes \begin{pmatrix} A_0^i \tilde{\partial\!\!\!/}_m A_1^i & \tilde{\gamma}_5 A_0^i (\xi B_1^i - A_1^i \xi) \\ \tilde{\gamma}_5 B_0^i (\xi^* A_a^i - B_1^i \xi^*) & B_0^i \tilde{\partial\!\!\!/}_n B_1^i \end{pmatrix} \right. \\
&\qquad \left. + \gamma_5 \otimes \begin{pmatrix} A_0^i \bar{\tilde{\partial\!\!\!/}}_m A_1^i & \tilde{\gamma}_5 A_0^i (\bar{\xi} B_1^i - A_1^i \bar{\xi}) \\ \tilde{\gamma}_5 B_0^i (\bar{\xi}^* A_a^i - B_1^i \bar{\xi}^*) & B_0^i \bar{\tilde{\partial\!\!\!/}}_n B_1^i \end{pmatrix} \right\} \\
&= \begin{pmatrix} A & \gamma_5 \phi \\ \gamma_5 \phi^* & B \end{pmatrix} + \gamma_5 \otimes \begin{pmatrix} C_A & \tilde{\gamma}_5 \beta \\ \tilde{\gamma}_5 \beta^* & C_B \end{pmatrix} + \gamma_5 \otimes \begin{pmatrix} \bar{C}_A & \tilde{\gamma}_5 \bar{\beta} \\ \tilde{\gamma}_5 \bar{\beta}^* & \bar{C}_B \end{pmatrix}.
\end{aligned}
\tag{33}
$$

We thus have ghosts and anti-ghosts for each gauge group $(C_{A,B}, \bar{C}_{A,B})$, gauge fields (A, B) and now matter fields, ϕ, with corresponding extensions to ghost and anti-ghost matter fields $(\beta, \bar{\beta})$ consistent with the interpretation of ϕ as an extended notion of gauge field. Note that the Hermiticity requirement on $\pi(\tilde{\omega})$ connects us with the Lie algebra for each group.

Proceeding as with Baulieu *et al* [13] we will now consider the generalized curvature, imposing the Cartan–Maurer condition to derive the constraints. Using the conditions (25) the complexity of this can be greatly reduced. We have the curvature at the level of the algebra

$$
\Theta = \tilde{\delta}\tilde{\omega} + \tilde{\omega}^2 \tag{34}
$$

where

$$
\begin{aligned}
\tilde{\delta}\tilde{\omega} &= (\delta + \delta_Q + \delta_{\bar{Q}}) \sum_i (a^i (\delta + \delta_Q + \delta_{\bar{Q}}) b^i) \\
&= \delta a^i \delta b^i + \delta a^i \delta_Q b^i + \delta a^i \delta_{\bar{Q}} b^i + \delta_Q a^i \delta b^i \\
&\quad + \delta_Q a^i \delta_Q b^i + \delta_Q a^i \delta_{\bar{Q}} b^i + \delta_{\bar{Q}} a^i \delta b^i + \delta_{\bar{Q}} a^i \delta_Q b^i \\
&\quad + \delta_{\bar{Q}} a^i \delta_{\bar{Q}} b^i
\end{aligned}
\tag{35}
$$

terms of the form $a^i \delta \delta_Q b^i$ vanishing. This illustrates the utility of working on the algebra where the calculus is well defined. Using the nomenclature of Baulieu *et al* [13] we set to zero the terms of $\pi(\Theta)$ proportional to $\mathrm{d}x^\mu \wedge \mathrm{d}y^p$, $\mathrm{d}x^\mu \wedge \mathrm{d}\bar{y}^p$, $\mathrm{d}y^p \wedge \mathrm{d}y^{p'}$, $\mathrm{d}y^p \wedge \mathrm{d}\bar{y}^{p'}$ and $\mathrm{d}\bar{y}^p \wedge \mathrm{d}\bar{y}^{p'}$, noting that y^p are coordinates on one of the two group manifolds. (Note, however, that we do not get mixed forms between groups, e.g. $\mathrm{d}y_m^p \wedge \mathrm{d}y_n^{p'}$.) Since matter fields are also present we will get forms proportional to $\mathrm{d}y^p$ and $\mathrm{d}\bar{y}^p$, corresponding to the covariant derivatives of the matter fields on the group manifold. These are also set to zero. We recall that such forms are still 2-forms in the generalized sense of Connes' construction [3–5]. Setting

$$
S_A = \gamma_m^p \partial_p \ \ (\leftrightarrow \mathrm{d}y_m^p\, \partial_p) \qquad S_B = \gamma_n^p \partial_p \ \ (\leftrightarrow \mathrm{d}y_n^p\, \partial_p) \tag{36}
$$

we derive the constraints:

$$
\begin{aligned}
S_A A^\mu &= D^\mu C_A & S_B B^\mu &= D^\mu C_B \\
S_A \phi &= -C_A(\phi + M) & S_B \phi &= (\phi + M) C_B \\
S_A \phi^* &= (\phi^* + M^*) C_A & S_B \phi^* &= -C_B(\phi^* + m^*) \\
S_A \beta &= -C_A(\beta + \xi) & S_B \beta &= (\beta + \xi) C_B \\
S_A \beta^* &= (\beta^* + \xi^*) C_A & S_B \beta^* &= -C_B(\beta^* + \xi^*) \\
S_A \bar{\beta} &= -C_A(\bar{\beta} + \bar{\xi}) & S_B \bar{\beta} &= (\bar{\beta} + \bar{\xi}) C_B \\
S_A \bar{\beta}^* &= (\bar{\beta}^* + \bar{\xi}^*) C_A & S_B \bar{\beta}^* &= -C_B(\bar{\beta}^* + \bar{\xi}^*) \\
S_A C_A &= -1/2[C_A, C_A] & S_B C_B &= -1/2[C_B, C_B]
\end{aligned}
\tag{37}
$$

and similarly for $\bar{S}_A$ and $\bar{S}_B$, for example,

$$
\begin{aligned}
&\bar{S}_A A^\mu = D^\mu \bar{C}_A && \bar{S}_B B^\mu = D^\mu \bar{C}_B \\
&\bar{S}_A \beta = -\bar{C}_A(\beta + \xi) && \bar{S}_B \beta = (\beta + \xi)\bar{C}_B \\
&\bar{S}_A \bar{C}_A = -1/2[\bar{C}_A, \bar{C}_A] && \bar{S}_B \bar{C}_B = -1/2[\bar{C}_B, \bar{C}_B] \quad \text{etc}
\end{aligned}
\tag{38}
$$

and the cross term

$$S_X \bar{C}_X + \bar{S}_X C_X + [\bar{C}_X, C_X] = 0 \tag{39}$$

where $X = A, B$. Note that we are implicitly working on the correct space of forms not having considered the differential ideal J. That is, we have not exhibited the auxiliary terms which arise in the curvature due to the ambiguity in the representation π[4]. These auxiliary terms are crucial for the correct determination of the Higgs potential but do not impact on the constraint equations.

As expected we reproduce the usual BRS and anti-BRS constraints on each gauge group. However, we now also have constraints involving the new field β. The element which is missing is the Zinn–Justin auxiliary field required to define $S_X \bar{C}_X$, which is left arbitrary by the constraints. Baulieu *et al* [13] introduce this field, b, so that

$$S\bar{C} = b \tag{40}$$

and so closing the algebra. Rather than introduce a new field we shall propose that the role of b is instead taken by β. This appears to be consistent as β should not appear in the physical equations of motion and thus must be treated as auxiliary. Furthermore, we now have a means by which the field b, defined in the adjoint of a given gauge group, may be decomposed into constituent parts. This is a natural consequence of dealing with representations on Hilbert space where fermionic elements are the basic building blocks.

To recover the Zinn–Justin auxiliary field we apply the constraints (37) so that (passing to the symmetric phase $\tilde{\beta} = \beta + \xi$)

$$\left.\begin{aligned} S_A\tilde{\beta} &= -C_A\tilde{\beta} \\ S_A\tilde{\beta}^* &= \tilde{\beta}^* C_A \end{aligned}\right\} \Rightarrow S_A(\tilde{\beta}\tilde{\beta}^*) = (S_A\tilde{\beta})\tilde{\beta}^* + \tilde{\beta} S_A \tilde{\beta}^* = [\tilde{\beta}\tilde{\beta}^*, C_A] \tag{41}$$

so that in this case $b = \tilde{\beta}\tilde{\beta}^*$. Similarly,

$$\left.\begin{aligned} S_B\tilde{\beta} &= \tilde{\beta} C_B \\ S_B\tilde{\beta}^* &= -C_B\tilde{\beta}^* \end{aligned}\right\} \Rightarrow S_B(\tilde{\beta}^*\tilde{\beta}) = (S_B\tilde{\beta}^*)\tilde{\beta} + \tilde{\beta}^* S_B \tilde{\beta} = [\tilde{\beta}^*\tilde{\beta}, C_B] \tag{42}$$

so for the B gauge field sector we have the identification $b = \tilde{\beta}^*\tilde{\beta}$. (An alternative is to identify the A and B gauge fields. The β fields are then necessarily in self-adjoint $m \times m$ representations of the gauge group [6]. It follows then from the constraints (37) that

$$S\tilde{\beta} = 1/2[\tilde{\beta}, C]. \tag{43}$$

The factor of $\frac{1}{2}$ is spurious since we should strictly make this identification at the level of the connection, not the constraints. This approach, however, is not a preferred option as it can be shown that then

$$\pi(\delta(\ker\pi)^1) = 0 \tag{44}$$

on the internal space and as a result one will no longer yield a Higgs potential in a symmetry-breaking form [18]). In our construction we also have the 'charge conjugate' field $\bar{\beta}$. We can, as above, construct the conjugate auxiliary field. The most general such term which we can construct has the form $b = (\tilde{\beta} + \tilde{\bar{\beta}})(\tilde{\beta} + \tilde{\bar{\beta}})^* + (\tilde{\beta} + \tilde{\bar{\beta}})^*(\tilde{\beta} + \tilde{\bar{\beta}})$. It follows from the constraints that

$$S_X(b) = [b, C_X] \qquad \bar{S}_X(b) = [b, \bar{C}_X]\,. \tag{45}$$

We now make the identification

$$S_X\bar{C}_X = b \qquad \bar{S}_X C_X = -b - [\bar{C}_X, C_X]\,. \tag{46}$$

Note that it is not necessary to keep track of which gauge sector we are considering since, for example,

$$\begin{aligned} S_A(\tilde{\beta}^*\tilde{\beta}) &= (S_A\tilde{\beta}^*)\tilde{\beta} + \tilde{\beta}^* S_A\tilde{\beta} \\ &= \tilde{\beta}^* C_A\tilde{\beta} - \tilde{\beta}^* C_A\tilde{\beta} \\ &= 0\,. \end{aligned} \tag{47}$$

Applying the nilpotency conditions (25), which on the correct space of forms are represented as

$$S_X^2 = \bar{S}_X^2 = 0 \qquad S_X\bar{S}_X + \bar{S}_X S_X = 0 \tag{48}$$

it follows that

$$S_X(b) = 0 \qquad \bar{S}_X(b) = [b, \bar{C}_X] \tag{49}$$

so that we have constrained $[b, C_X] = 0$ by this choice. We can thus recover, in a very natural way, the set of BRS and anti-BRS equations on each gauge group where now the matter field constraints on ϕ arise at the same level as those of the gauge fields. The Zinn–Justin auxiliary scalar now appears as a result of this fundamental Higgs-gauge field unification rather than as an additional field. To insure the auxiliary nature of β we also impose the constraint

$$D_\mu\beta = 0\,. \tag{50}$$

That β should be trivial on space-time is consistent with the notion that it is a connection between group manifolds only. (It is tempting to try a more symmetric identification than (46) and extend the choice of Baulieu *et al* [13] such that

$$S_X b = 0 \qquad \bar{S}_X = [b, \bar{C}_X] \tag{51}$$

now includes

$$S_X b^c = [b^c, C_X] \qquad \bar{S}_X b^c = 0 \tag{52}$$

where $b = \tilde{\beta}^*\tilde{\beta} + \tilde{\beta}\tilde{\beta}^*$ and $b^c = \tilde{\bar{\beta}}^*\tilde{\bar{\beta}} + \tilde{\bar{\beta}}\tilde{\bar{\beta}}^*$. This would extend the construction of Baulieu *et al* [13] to include b^c in such a way that b^c is a constant of the motion in the anti-ghost sector in a complimentary way to b on the ghost sector. However, it is not possible to make this identification and keep compatability with the remaining algebra, for example, requiring that $S_X^2\bar{C}_X = 0$. For this reason the choice of (46) is the appropriate one).

While this is sufficient when restricted to the gauge fields, the existence of Higgs scalars connecting different gauge fields requires independent consideration. This is because generalized 2-forms in the construction of Connes' [3–5] will now arise which are scalars in the traditional sense. This is just the origin of the Higgs potential in Connes model building prescription. However, we now also have β and $\bar{\beta}$ terms which will contribute. We thus expect to generalize our potential, which will take the form

$$V(\phi, \beta, \bar{\beta}) = V(\phi) + V(\beta, \bar{\beta}) + \text{mixing terms}\,. \tag{53}$$

The mixing terms demand that β and $\bar{\beta}$ remain the fundamental auxiliary fields, rather than b. Elimination of β and $\bar{\beta}$ will now have a dramatic effect, introducing additional interaction terms between the Higgs, ghost and gauge fields. The actual form taken by the interaction terms will be very much model-dependent and so will be investigated in specific examples currently in preparation. The important consequence of this is that we have extended the notion of gauge fixing into the Higgs sector, consistent with gauge-Higgs field unification.

6. The quantum Lagrangian

We now wish to consider the most general allowable BRS and anti-BRS invariant Lagrangian from which the BRS/anti-BRS admissable gauges may be considered. Again, restricting ourselves at first to the underlying algebra will greatly simplify this construction. We know that the form of our Higgs potential before elimination of the auxiliary terms derives from the square norm of the curvature. In looking at the quantum Lagrangian we wish to avoid the introduction of terms like ϕ^2 with arbitrary coefficients which are not forbidden by BRS/anti-BRS invariance but which could spoil the Higgs potential. We therefore choose to maintain the action as a functional of the curvature [1]. The simplest such term which can be constructed, which is δ_Q and $\delta_{\bar{Q}}$ invariant and of dimension 4, is

$$\delta_Q\delta_{\bar{Q}}\Theta = \delta_Q\delta_{\bar{Q}}(\tilde{\delta}\tilde{\omega} + \tilde{\omega}^2)\,. \tag{54}$$

To be physical we require the ghost number to be zero, insuring that we are in the correct cohomology class. Assigning a ghost number of 1, say, to δ_Q and -1 to $\delta_{\bar{Q}}$ we see that this restriction reduces (54) to

$$\delta_Q\delta_{\bar{Q}}\left(\delta\omega + \omega^2 + \alpha\sum_{ij}\left(a^i\delta_Q b^i a^j \delta_{\bar{Q}} b^j + a^i\delta_{\bar{Q}} b^i a^j \delta b^j\right)\right) \tag{55}$$

where $\omega = \sum_i a^i\delta b^i$ is the usual, classical, connection used in Connes' construction [5]. Here α may be introduced as an arbitrary constant, the choice of which corresponds to the gauge choice [13]. Note that these requirements on ghost number and δ_Q and $\delta_{\bar{Q}}$ invariance are imposed at the well behaved level of the algebra. Representing this on Hilbert space will, however, introduce terms with non-zero ghost number and which are not invariant under S_X and $\bar{S}_X$. This results from treating BRS/anti-BRS invariance in a unified way on the algebra but then separating contributions on the two gauge groups when physical fields are constructed. That the physical set of operators is a restricted set of those on the differential algebra is a recurring theme in all applications of non-commutative geometry.

Representing this on Hilbert space now requires some care. The reason is that contributions such as $\sum_i a^i\delta\delta_Q b^i$ need not vanish due to a lack of cross terms. To represent such terms the representation π must be extended to accommodate our generalized Leibnitz rule. We will thus define

$$\pi(\delta_Q\delta a^i) = \{Q, [D, \rho(a^i)]\} \qquad \pi(\delta_{\bar{Q}}\delta a^i) = \{\bar{Q}, [D, \rho(a^i)]\} \tag{56}$$

and

$$\pi(\delta_Q\delta_{\bar{Q}}\delta a^i) = [Q, \{\bar{Q}, [D, \rho(a^i)]\}] \tag{57}$$

which now encodes correctly the Leibnitz rule on each copy of space-time and on the matrix derivative connecting space-times.

To extract the physical fields from (55) we note that the action in non-commutative geometry which is given by the Dixmier trace can be written equivalently as [6]

$$I = \tfrac{1}{8}\int \mathrm{d}^4x\,\mathrm{Tr}(\mathrm{tr}(\pi^2(\theta))) \tag{58}$$

where θ is the usual curvature in Connes' construction ($\theta = \delta\omega + \omega^2$), Tr is taken over the matrix structure and tr is taken over the Clifford algebra. We extend this now to include the 'quantum term'

$$I_{\text{quantum}} = \tfrac{1}{8} \int \mathrm{d}^4x\, \mathrm{Tr}(\mathrm{tr}[\pi^2(\theta') + \pi(\delta_Q \delta_{\bar{Q}} \Theta)_{\text{zero ghost no}}]) \tag{59}$$

where we write θ' since we must also now accommodate the extended potential (53). We need thus consider only those terms which survive under the two trace operations. Note, however, that we will take the same liberty as Baulieu *et al* [13] and retain the fields C_X and $\bar{C}_X$ as well as S_X and $\bar{S}_X$ as differential forms. To this extent we only take tr over the four-dimensional space-time Clifford algebra. In terms of Dixmier traces we see then that the action need only be defined with respect to the classical K-cycle of continuous functions on space-time tensored with a discrete internal space. There is, therefore, no problem with (d, ∞) summability, consistent with imposing the Cartan–Maurer condition.

The simplest expression to calculate is $\pi(\delta_Q \delta_{\bar{Q}} \delta\omega)$ for which we find (retaining the S_X and $\bar{S}_X$ invariant terms)

$$S_A \bar{S}_A (M\phi^* + \phi M^*) + S_B \bar{S}_B (M^*\phi + \phi^* M) \tag{60}$$

where since we require global gauge invariance terms of the form $S_A \bar{S}_A (\partial_\mu A^\mu)$ have been excluded. Similarly for $\pi(\delta_Q \delta_{\bar{Q}} \omega^2)$ the result is

$$S_A \bar{S}_A (A_\mu^2 + \phi\phi^*) + S_B \bar{S}_B (B_\mu^2 + \phi^*\phi) \tag{61}$$

where we again note that since we require global gauge invariance $\phi\phi^*$ and $\phi^*\phi$ must transform as singlets. Consequently, $S_A \bar{S}_A (\phi\phi^*) = S_B \bar{S}_B (\phi^*\phi) = 0$. Greater care must be taken in determining the final term to avoid contributions with non-zero ghost number. Retaining the S_X and $\bar{S}_X$ invariant terms we find, after lengthy algebra, the relevant contributions for $\pi(\delta_Q \delta_{\bar{Q}} [\sum_{ij} (a^i \delta_Q b^i a^j \delta_{\bar{Q}} b^j + a^i \delta_{\bar{Q}} b^i a^j \delta_Q b^j)])$:

$$S_A \bar{S}_A (C_A \bar{C}_A + \bar{C}_A C_A + \beta\bar{\beta}^* + \bar{\beta}\beta^*) + S_B \bar{S}_B (C_B \bar{C}_B + \bar{C}_B C_B + \beta^*\bar{\beta} + \bar{\beta}^*\beta) \tag{62}$$

where we eliminate the β terms analogously to the ϕ's in (61). The removal of these terms is consitent with Hermiticity of the 'quantum term' as implied by (46). In all these expressions the auxiliary fields associated with the representation π have been suppressed. Being at fourth order, such auxiliary contributions are tediously complex so there is little utility in expressing their general form. We see then that we are left with only the most obviously S_X and $\bar{S}_X$ invariant terms as physical contributions. Clearly, a projection onto the correct space of forms will involve greater mathematical rigour and complexity. Nevertheless, we would expect the general form of the above expressions to be maintained since the requirement of global gauge invariance has removed terms dependent on ϕ and β for which we would expect non-trivial contributions.

We can thus recover, in a very natural way, the quantum term of Baulieu *et al* [13] in the case of symmetric gauges. Rather than an exhaustive construction of all possible allowable polynomials of the fields we can develop the same result directly from our generalized curvature, consistently including the contribution from the Higgs sector.

7. Application to anomalies

The constructions of the previous sections demonstrate that significant simplifications in the construction of field-theoretic models arise by first restricting considerations to the underlying universal differential algebra. This highlights the quantity of relevant information contained at this level of the model building prescription. Only when a particular representation, π, is chosen do we encounter complications. This is a reflection of the wide and as yet non-unique choice of representation on the Hilbert space of spinors.

Using this observation we can direct our attention to the important topic of anomalies. Since anomaly cancellations are dependent on the choice of fermionic representations and not fermion masses we see that we require only that information which is entered at the level of the algebra $\mathcal{A}_t$. Strictly, we should be dealing with a local BRS operator but this extension is not considered to be problematic.

It is known that consistency conditions for anomalies appear as cohomological equations for the BRS operator (we will neglect the anti-BRS operator for simplicity). In this regard we will utilize the trivial cohomological structures on the universal differential algebra to simplify considerations for otherwise complex models involving multiple gauge fields, Higgs scalars, etc. We will follow the prescription of Dubois-Violette *et al* [19] for constructing solutions of the Wess–Zumino consistency conditions [20]. We consider the free-graded commutative algebra generated by

$$\begin{aligned} A &= \sum_i a^i \delta b^i \qquad & F &= \delta A + A^2 \\ \chi &= \sum_i a^i \delta_Q b^i \qquad & \phi &= \delta \chi \end{aligned} \tag{63}$$

which we denote as C. A and χ comprise the generalized potential

$$\tilde{A} = \sum_i a^i (\delta + \delta_Q) b^i = A + \chi\,. \tag{64}$$

The generalized curvature is then

$$\tilde{F} = (\delta + \delta_Q)\tilde{A} + \tilde{A}^2\,. \tag{65}$$

Imposing the Maurer–Cartan condition at the level of the universal differential algebra it follows that

$$\delta_Q \chi = -\chi^2 \qquad \delta_Q A = -\phi - A\chi - \chi A \tag{66}$$

and therefore $\delta_Q F = [F, \chi]$. Note that as before, when considering the quantum Lagrangian, this BRS algebra is *not* equivalent to that derived on the physical Hilbert space. This follows since the above relations (66) imply that those terms dependent on β will not appear in the generalized potential (53). From this we see that the auxiliary nature of β has been made manifest. Consistent with Connes' model building prescription β remains as part of the generalized potential due only to the nature of the representation on Hilbert space. In this context, the removal of β via the equations of motion is motivated by purely algebraic considerations and not only phenomenological consistency. This Zinn–Justin scalar has thus been reduced to one of several auxiliary terms inherent in the model building scheme.

Using $A, \delta A, \chi$ and $\delta\chi$ as a free system of generators of C it follows that the algebra (C, δ) is a contractable differential algebra. Similarly, $A, \chi, (\delta + \delta_Q)A, (\delta + \delta_Q)\chi$ is a free system of generators of C so $(C, \delta + \delta_Q)$ is contractable. As can be seen from the BRS relations (66) the cohomology of δ_Q is not trivial. It follows that (C, δ_Q) is the skew tensor product of the contractable algebra $A, \delta_Q A$ and the algebra χ, F. The δ_Q cohomology thus reduces to that on χ and F. From this it can be shown that the δ_Q cohomology reduces to sets of invariant polynomials in these fields. Thus anomalies and Schwinger terms are obtained from such invariants [19].

This type of analysis remains faithful for simple gauge theories. But we now have the additional step of identifying with complex models involving multiple gauge fields and

Higgs scalars via the representation π. On the physical space such a simple cohomological treatment need not hold. However, as we saw in the previous section, representations of appropriate polynomial functions constructed on the universal differential algebra compactly describe all relevant polynomial contributions, including scalar contributions, on the physical space. There was no need to exhaustively explore all possible polynomial functions. Similarly, representing invariant polynomial functions pertaining to anomalies, which are simply described on the universal differential algebra, will encode all the relevant information on these contributions on the physical space without the need for an arbitrary exhaustive search; the relevant physical terms being extracted as before by such requirements as S_X or global gauge invariance. This provides an important tool for probing complicated models. We would still expect model dependency in this approach as the δ_Q cohomology is dependent on the number of $U(1)$ factors in the model in question.

8. Some comments on auxiliary terms

The astute reader will observe that the gauge terms appearing in (61) are normally considered as auxiliary in Connes' construction [5] but are retained in the 'quantum' sector of our action (59). This is not inconsistent when we examine the nature of these auxiliary terms more closely.

That problems arising from such contributions can be seen if one considers the space of generalized 2-forms in the classical case [7]

$$\hat{\Omega}^2(\mathcal{A}_t) = [\hat{\Omega}^2(\mathcal{F}) \otimes \rho(\mathcal{A}) + \mathcal{F} \otimes \hat{\Omega}^2(\mathcal{A})] \oplus \hat{\Omega}^1(\mathcal{F}) \otimes \hat{\Omega}^1(\mathcal{A}) \tag{67}$$

where we recall that $\mathcal{A}_t = \mathcal{F} \otimes \mathcal{A}$, corresponding to the product geometry of smooth functions on space-time and a discrete internal space. The first term gives the usual gauge field curvature tensor, the second corresponds to the square root of the Higgs potential and the last is responsible for the covariant derivative on the matter fields. The addition in brackets is not direct because space-time 0-forms and 2-forms mix. Consequently, in the tensor product geometry, the Higgs potential contributions must be disentangled from the 0-forms of the gauge sector. This is the fundamental reason for the need of at least two fermionic families to ensure non-trivial projections onto the correct orthocomplement. We can write down these 0-form contributions explicitly in, say, the 'A' gauge field sector, which are [6]:

$$\sum_i A_0^i \partial\!\!\!/^2 A_1^i + \partial^\mu A_\mu + A_\mu^2 \,. \tag{68}$$

The first appears when one calculates $\pi(\sum_i \delta a^i \delta b^i)$ as an element which cannot be expressed in terms of the physical fields while the remaining two arise from the Clifford algebra, e.g.

$$\partial\!\!\!/ \gamma^\nu A_\nu = 1/2\sigma^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) - \partial^\mu A_\mu \tag{69}$$

where $\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}$ in four-dimensional Euclidean space. Thus $\partial^\mu A_\mu$ is associated with $\pi(\delta\rho)$ while similarly A_μ^2 derives from $\pi(\rho^2)$. The important point, however, is that since $\sum_i A_0^i \partial\!\!\!/^2 A_1^i$ is an arbitrary function the terms of the scalar Higgs potential could be absorbed into it. Thus it is this term which is responsible for the need for careful consideration on the correct space of forms. The remaining terms are, nevertheless, unsavoury, implying that space-time 2-forms in Connes construction will not correspond to our usual notion of 2-form unless eliminated.

This brings us now to the 'quantum term' $\pi(\delta_Q \delta_{\bar{Q}} \Theta)_{\text{zero ghost no}}$ of our action. Unlike in (58), this term is not quadratic in the curvature. Immediately this implies that terms such as

A_μ^2 will play a more fundamental role, being Lorentz scalars. It seems natural then to treat $\pi(\delta_Q \delta_{\bar{Q}} \Theta)_{\text{zero ghost no}}$ independently to the classical action, a point emphasized by it being dependent on S_X and $\bar{S}_X$, unlike the classical case. We thus confront the interesting point of treating as auxiliary the fields $\partial^\mu A_\mu + A_\mu^2$ whose elements are well defined in terms of known fields and do not otherwise adversely affect the action. In our context they find a role which need not violate our usual notion of differential forms when treated independently from them. This notion is consistent with the imposition of the Maurer–Cartan form in deriving the BRS/anti-BRS constraints which can be considered as a 'horizontality condition'. The 'quantum' contribution then comes from the vertical, orthogonal sector generated by non-vanishing terms in the BRS/anti-BRS generators. An interesting implication of this is that the space-time Dirac operator in the vertical sector will no longer be nilpotent, implying that higher-order terms will be unphysical.

Turning to the BRS and anti-BRS region we would expect generalized 2-forms to encounter the same problems as the Dirac operator, D, as expressed in (67). Since β-type terms will contribute to the scalar potential these, as with the ϕ scalars, will necessarily carry information pertaining to several fermionic families. Consistency demands that these family mixing matrices be identical for ϕ and β. In addition to this we must also account for the nilpotency conditions encoded in (25) between the operators δ, δ_Q and $\delta_{\bar{Q}}$. That is, there will exist forms for which, for example,

$$\pi(\delta_Q \delta_{\bar{Q}} a^i + \delta_{\bar{Q}} \delta_Q a^i) \neq 0 . \tag{70}$$

This simply tells us that auxiliary terms will arise in accordance with mixing terms, as expressed in the potential (53). (Note that, by the definition of our generalized Dirac operator $\tilde{D}$ (31), there will be no such problem between $\begin{pmatrix}\not\partial & \\ & \not\partial\end{pmatrix}$ and Q or $\bar{Q}$ so that in this case only the connection between space-times and group manifolds will yield new contributions to J. This, however, will not be the case between Q and $\bar{Q}$. Evidence for additional structure introduced by Q and $\bar{Q}$ on the discrete space was demonstrated when it became obvious that many new auxiliary terms were arising due to the existence of ξ and $\bar{\xi}$, which would otherwise vanish if we imposed $\xi = \bar{\xi}$.) The model dependency of the scalar potential is thus further emphasized by these considerations.

9. Conclusion

By utilizing the underlying mathematical approach of Connes construction we have compactly described the BRS/anti-BRS structure of complicated models involving multiple gauge and Higgs fields. Furthermore, we have proposed a natural origin to the Zinn–Justin auxiliary scalar consistent with the algebraic structure of non-commutative geometry. That non-commutative geometry is indeed a natural arena for this type of investigation is emphasized when it is recalled that fermionic fields, having a canonical dimension of $\frac{3}{2}$, will not contribute to the 'quantum term' of the action. The fermionic sector can, as in the classical case, be introduced in a trivial way.

We have not explicitly considered the possibility of assymetric gauge choices corresponding to breaking hermiticity, as implied by (46). Actually, we note that the potential (53) implies that this hermiticity condition is already broken. Terms do arise in the calculation of $\pi(\delta_Q \delta_{\bar{Q}} \Theta)_{\text{zero ghost no}}$ which appear to fill this role, however, these are not $\bar{S}_X$ invariant. This is actually a good result for two reasons:

(i) such terms are associated with other terms for which zero ghost number fails,

(ii) to explore the full range of possible gauges a new gauge parameter is required, implying the introduction of an additional 'quantum term'.

One such possible term which suggests itself is $\pi(\delta_Q\tilde{\omega}\Theta)_{\text{zero ghost no}}$, allowing us to maintain the action as a functional of the curvature. This does not obviously introduce BRS/anti-BRS invariant terms so that the naturalness of such an expression needs to be tested.

An interesting point which remains is the interpretation of β. One natural possibility is that ξ encodes information on the relative strengths of the group manifolds. That is, we *have no fixed notion of the relative sizes of these internal spaces. In this sense ξ contains* information on the relative coupling strengths. A topologically more appealing possibility is that ξ connects different choices of gauge on the different group manifolds. The true nature of this perhaps waits for a full quantum treatment. This is, of course, speculation, but signals the possibility of interesting new contributions for a non-commutative quantum field theory.

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